Pergamon

Int. J. Solids Structures Vol. 35, Nos. 3–4, pp. 299–318, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00073–5

THREE-DIMENSIONAL ELASTICITY SOLUTIONS FOR FREE VIBRATIONS OF RECTANGULAR PLATES BY THE DIFFERENTIAL QUADRATURE METHOD

MOINUDDIN MALIK and CHARLES W. BERT
School of Aerospace and Mechanical Engineering, The University of Oklahoma, 865 Asp Avenue, Room 212, Norman, OK 73019-0601, U.S.A.

(*Received* 3 *September* 1996; *revised form* 24 *February* 1997)

Abstract—This paper presents accurate three-dimensional elasticity solutions for free vibrations of six types of plates having free lateral surfaces, two opposite sides simply supported, and two other sides having combinations of simply supported, clamped, and free boundary conditions. The solution methodology adopted in the present work first reduces the spatial variables in the governing elasticity equations to two by using displacement functions in the forms which satisfy the boundary conditions of the assumed simply supported sides. The eigenvalue equations are then formulated from the reduced governing equations and the boundary conditions via the differential quadrature method. The numerical results include the first nine natural frequencies of the six plate configurations for combinations of three aspect ratios and two thickness ratios. These results are supported by appropriate convergence studies and comparisons with the results of other authors. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Two-dimensional theories, such as the classical thin plate theory (CPT), first-order shear deformable plate theory (FSDPT), and the higher-order shear deformable plate theories (HSDPT's), are mechanics of materials and applied elasticity approaches for the static and dynamic flexure of plates. These theories offer in most situations, if not all, accurate and reliable solutions for the analysis and design of plates and plate systems. However, three-dimensional elasticity analysis of the plate problems has attracted quite a number of researchers.† A three-dimensional analysis does not rely on any hypotheses concerning the kinematics of deformation and, therefore, such analyses not only provide realistic results but also bring out physical characteristics which can not otherwise be predicted by two-dimensional analyses. Furthermore, three-dimensional elasticity solutions provide a real basis for assessing the results of the two-dimensional theories.

Looking into the literature, one finds that the available three-dimensional elasticity solutions of rectangular plates are very limited with respect to the boundary conditions. The application of calculus of variations to the energy functional of three-dimensional elasticity leads to as many as eight sets of a triplet of boundary conditions at each side of a plate. Thus, the number of plate configurations based on mathematically plausible combinations of boundary conditions on the six faces of a plate would be extremely large. The boundary conditions of practical interest are the ones in which top and bottom (horizontal) surfaces are free and on each vertical side, there is one of the simply supported, clamped, or free conditions. Such conditions give rise to 21 plate configurations. However, even with this reduced size, the vibration analysis through three-dimensional elasticity is a formidable task. The reason for this is that analytical solutions are possible only for some specific cases and, in general, analysis has to be carried out numerically.

A set of plate configurations which offers advantage in analytical and semi-analytical solutions is one having two opposite sides simply supported; in those cases the three displacement components can be represented by trigonometric sine and cosine functions of

† Since such analyses do not impose any restriction on the thickness, in the related works, thick plates have also been referred to as parallelepipeds.

the coordinate direction perpendicular to the simply supported sides. Such a representation satisfies identically the boundary conditions of the simply supported sides and results in reduction of one spatial variable from the governing differential equations and the remaining boundary conditions as the trigonometric terms cancel out from both sides of the equations.† In the case of a plate simply supported on all four sides (the SSSS plate),‡ the governing partial differential equations are actually reduced to ordinary differential equations with respect to the through-thickness spatial variable; the reduced equations may then be solved analytically or numerically.

Among the solutions of simply supported plates, one comes across the works of Pagano (1969, 1970) who considered static bending of infinitely long and finite size composite laminates under sinusoidal lateral loadings. In these works the elasticity solutions were compared with the CPT solutions and the limitations of the latter theory were pointed out. Srinivas *et al.* (1970a, b, c) presented a very complete work on bending, vibration, and buckling analyses of plates of both isotropic and orthotropic materials. These works compared the elasticity solutions with the CPT and FSDPT solutions. The most important contribution of these works was the identification of certain vibration and buckling modes which could be obtained from the elasticity solutions alone. Based on the analysis of Srinivas *et al.*, Wittrick (1987) worked out a detailed analytical investigation of the elasticity solution of simply supported plates for eigenvalue problems of buckling and free vibration and for static deflection under sinusoidal lateral loading. Another significant contribution is the work of Noor (1973a, b; 1975) who used an efficient finite difference scheme for the solution of reduced ordinary differential equations for vibration and buckling analyses of simply supported plates comprising of a large number of orthotropic layers.

Early efforts on numerical vibration analysis through three-dimensional elasticity equations include the work of Cheung and Chakrabarti (1972); these authors used the finite layer method and obtained natural frequencies of five different plate configurations (CCCC, CCCF, SCCS, SCSC, and SSSC plates). Iyengar and Raman (1980) applied the method of initial functions to the problem and reported frequencies of SCSC and CCCC plates. In their work, Cheung and Chakrabarti (1972) assessed the accuracy of their results on the basis of comparisons with the results of Srinivas *et al.* (1970a) for frequencies of simply supported plates. Iyengar and Raman (1980) compared their results with only those of Cheung and Chakrabarti (1972). In the absence of results for comparison and even otherwise, numerical results need to be supported by appropriate convergence analyses which often provide a sound basis for assessing the accuracy of results; both of these works did not include analyses of this kind. In this regard, recent works of Liew *et al.* (1993, 1994, 1995) are highly commendable in that comprehensive investigations were carried out and detailed results were presented for frequencies and mode shapes of SSSS, SCSC, SFSF, and CCCC plates. In these works, the solution method is based on three-dimensional Ritz formulation with orthogonal polynomials and convergence analyses have been carried out in sufficient detail. The results appear to be of very reliable accuracy.

Among other solutions, mention may also be made of the elasticity solutions of the 'complicated' fully free (FFFF) and cantilevered (CFFF) plates. Fromme and Leissa (1970) presented an exact analysis for the vibration of FFFF plates using an extended Fourier series method (referred to as the method of associated periodicity); the numerical results were, however, reported for infinitely long plates only. The method of analysis of Hutchinson and Zillmer (1983) was based on series solutions for all possible modes of vibration of FFFF plates of finite dimensions. Very recently, Young *et al.* (1996) have considered the vibration problem of FFFF plates with depressions, grooves or cut-outs; the solution was obtained by the Ritz method using polynomials as trial functions. The elasticity solution for vibration of CFFF plates was presented by Leissa and Zhang (1983). The only other work on such plates is that of Liew *et al.* (1993) in which a table is provided comparing

† This is indeed a well known general features which is also advantageous in classical thin and shear deformable analyses of rectangular plates.

‡ A standard notation wherein plates are designated by four letters ordered to indicate the boundary conditions of sides $x = 0$, $y = 0$, $x = a$, and $y = b$; see Fig. 1. The letters S, C, and F denote simply supported, clamped, and free conditions, respectively.

these authors' calculated frequencies with those of Leissa and Zhang (1983). It needs to be mentioned here that some work is also available on elasticity solutions of twisted cantilevered panels; however, this is unrelated to the present work which is concerned with only initially flat rectangular panels.

The present work concerns three-dimensional elasticity solutions for vibration analysis of plate configurations having two opposite (vertical) sides simply supported and general boundary conditions at the other two sides. The interest in this work comes from the fact that, as mentioned earlier, by having two opposite sides simply supported, numerical solutions may be handled in semi-analytical manner. In fact, this advantage has been taken by Mizusawa and Takagi (1995) by employing trigonometric solutions between the two opposite simply supported sides in a semi-analytical solution for such plates using the spline prism method. By having combinations of simply supported, clamped, and free boundary conditions at the other two sides, one can have six plate configurations and, the work of Mizusawa and Takagi (1995) is possibly the only one in which all six cases have been considered in totality. However, it is noted that, besides the exact solutions of SSSS plates, some of these configurations have been considered in earlier mentioned works (Cheung and Chakrabarti, 1972; Iyengar and Raman, 1980; Liew *et al.*, 1993, 1994, 1995) as well.

The present work offers an alternate solution method for the three-dimensional elasticity analysis for vibration of plate configurations of interest. In the following description, advantage is taken of having the two opposite sides simply supported to reduce the number of spatial variables in the governing elasticity equations to two. The solution of the reduced two-dimensional equations is then obtained via the differential quadrature method (DQM). In this paper, details of the DQM are not given; interested readers may refer to the works of Bellman *et al.* (1971, 1972) who originated the method and a recent survey paper of the present investigators (Bert and Malik, 1996a). However, the following analysis includes quadrature formulation of the reduced two-dimensional equations and their boundary conditions. Also included in the following analysis is the special case of the pure shear mode of vibration. In this case, the governing equations become decoupled and may be solved to obtain exact closed-form frequency equations for all cases of boundary conditions considered in the present work.

The numerical results include the first nine vibration frequencies of the six plate configurations for combinations of three aspect ratios and two thickness ratios. Of course, these results are supported by convergence studies and comparisons with the results of other investigators.

2. GOVERNING ELASTICITY EQUATIONS

Consider an isotropic-material plate (or a parallelepiped), Fig. 1, bounded by the coordinate planes $x = 0$, a, $y = 0$, b, and $z = 0$, h. The governing elasticity equations of

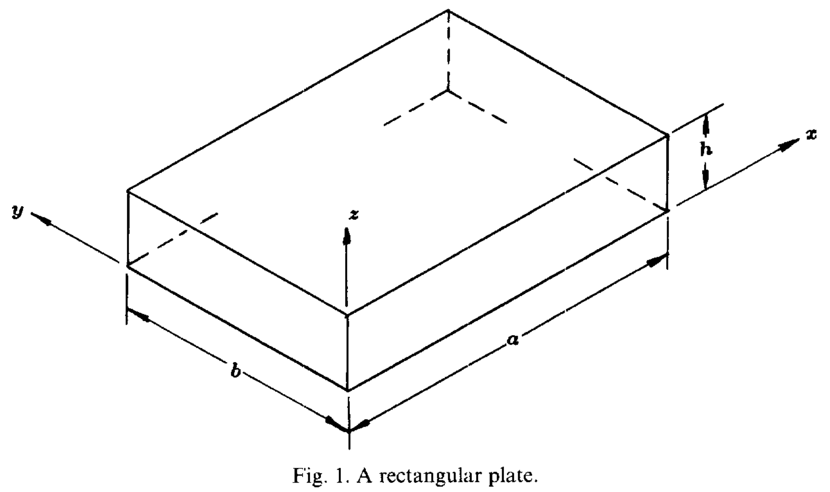

Fig. 1. A rectangular plate.

the plate undergoing simple harmonic oscillations in a principal vibratory mode are written in a nondimensional form as (Srinivas *et al.*, 1970a)

$$\nabla^2(U,V,W)+\frac{1}{1-2\nu}\left(\frac{\partial}{\partial X},\lambda\frac{\partial}{\partial Y},\alpha\frac{\partial}{\partial Z}\right)\Theta = -\Omega^2(U,V,W) \tag{1}$$

where $(U, V, W) = (u, v\,w)/h$ are the dimensionless displacement components in the x, y, and z directions, respectively; $(X, Y, Z) = (x/a, y/b, z/h)$ are the normalized coordinates; $\alpha = a/h$ is the lateral aspect ratio; $\lambda = a/b$ is the in-plane aspect ratio; ν is the Poisson's ratio;

$$\nabla^2 = \frac{\partial^2}{\partial X^2}+\lambda^2\frac{\partial^2}{\partial Y^2}+\alpha^2\frac{\partial^2}{\partial Z^2}$$

is the Laplacian operator;

$$\Theta = \frac{\partial U}{\partial X}+\lambda\frac{\partial V}{\partial Y}+\alpha\frac{\partial W}{\partial Z}$$

is the volumetric dilatation; and Ω is the dimensionless frequency

$$\Omega^2 = \frac{2(1+\nu)\rho a^2}{E}\omega^2$$

in which ω is the circular frequency (rad/s); and E and ρ are, respectively, the elastic modulus and density of plate material.

Let the plate be simply supported on the sides $x = 0$, a, i.e., $X = 0$, 1; the boundary conditions on these sides are prescribed by the normal traction and the tangential displacements being equal to zero so that

$$\frac{\partial U}{\partial X} = 0, \quad V = 0, \quad W = 0 \quad \text{at } X = 0, 1. \tag{2}$$

The displacement components satisfying the boundary conditions on the x-sides, eqn (2), may be expressed as

$$U = \bar{U}(Y,Z)\cos\bar{m}X, \quad V = \bar{V}(Y,Z)\sin\bar{m}X, \quad W = \bar{W}(Y,Z)\sin\bar{m}X \tag{3}$$

where $\bar{m} = m\pi$ and m is an integer.

Substituting eqn (3) in eqn (1), one can express the governing equations in terms of reduced displacement variables $\bar{U}$, $\bar{V}$, and $\bar{W}$ as

$$-\left\{\lambda^2\frac{\partial^2}{\partial Y^2}+\alpha^2\frac{\partial^2}{\partial Z^2}-\frac{2(1-\nu)}{1-2\nu}\bar{m}^2\right\}\bar{U}-\frac{\bar{m}\lambda}{1-2\nu}\frac{\partial\bar{V}}{\partial Y}-\frac{\bar{m}\alpha}{1-2\nu}\frac{\partial\bar{W}}{\partial Z} = \Omega^2\bar{U}$$

$$\frac{\bar{m}\lambda}{1-2\nu}\frac{\partial\bar{U}}{\partial Y}-\left\{\frac{2(1-\nu)}{1-2\nu}\lambda^2\frac{\partial^2}{\partial Y^2}+\alpha^2\frac{\partial^2}{\partial Z^2}-\bar{m}^2\right\}\bar{V}-\frac{\alpha\lambda}{1-2\nu}\frac{\partial^2\bar{W}}{\partial Y\partial Z} = \Omega^2\bar{V}$$

$$\frac{\bar{m}\alpha}{1-2\nu}\frac{\partial\bar{U}}{\partial Z}-\frac{\alpha\lambda}{1-2\nu}\frac{\partial^2\bar{V}}{\partial Y\partial Z}-\left\{\lambda^2\frac{\partial^2}{\partial Y^2}+\frac{2(1-\nu)}{1-2\nu}\alpha^2\frac{\partial^2}{\partial Z^2}-\bar{m}^2\right\}\bar{W} = \Omega^2\bar{W}. \tag{4}$$

It is noted that the original three-dimensional elasticity equations, eqn (1), have now been reduced to two-dimensional equations described in a normalized square domain $0 \leqslant Y \leqslant 1$, $0 \leqslant Z \leqslant 1$.

The sides $y = 0$, b may be either simply supported, clamped, or free. At a simply supported y-side, with zero normal traction and zero tangential displacements, the boundary conditions in terms of reduced displacement variables become

$$\bar{U} = 0, \quad \frac{\partial \bar{V}}{\partial Y} = 0, \quad \bar{W} = 0 \quad \text{at } Y = 0 \text{ and/or } 1. \tag{5}$$

At a clamped y-side, $u = 0$, $v = 0$, and $w = 0$ so that

$$\bar{U} = 0, \quad \bar{V} = 0, \quad \bar{W} = 0 \quad \text{at } Y = 0 \text{ and/or } 1. \tag{6}$$

At a free y-side, the conditions of components of traction equal to zero may be written as

$$\lambda\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = 0, \quad \frac{\partial U}{\partial X} + \frac{1-\nu}{\nu}\lambda\frac{\partial V}{\partial Y} + \alpha\frac{\partial W}{\partial Z} = 0, \quad \alpha\frac{\partial V}{\partial Z} + \lambda\frac{\partial W}{\partial Y} = 0, \quad \text{at } Y = 0 \text{ and/or } 1$$

and using eqn (3), these conditions become

$$\lambda\frac{\partial \bar{U}}{\partial Y} + \bar{m}\bar{V} = 0, \quad \bar{m}\bar{U} - \frac{1-\nu}{\nu}\lambda\frac{\partial \bar{V}}{\partial Y} - \alpha\frac{\partial \bar{W}}{\partial Z} = 0, \quad \alpha\frac{\partial \bar{V}}{\beta Z} + \lambda\frac{\partial \bar{W}}{\partial Y} = 0 \quad \text{at } Y = 0 \text{ and/or } 1. \tag{7}$$

The lateral surfaces of the plate are assumed to be free so that the traction components are zero on $z = 0$, h. These conditions may be written as

$$\alpha\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} = 0, \quad \alpha\frac{\partial V}{\partial Z} + \lambda\frac{\partial W}{\partial Y} = 0, \quad \frac{\partial U}{\partial X} + \lambda\frac{\partial V}{\partial Y} + \frac{1-\nu}{\nu}\alpha\frac{\partial W}{\partial Z} = 0 \quad \text{at } Z = 0 \text{ and/or } 1$$

which, on using eqn (3), become

$$\alpha\frac{\partial \bar{U}}{\partial Z} + \bar{m}\bar{W} = 0, \quad \alpha\frac{\partial \bar{V}}{\partial Z} + \lambda\frac{\partial \bar{W}}{\partial Y} = 0, \quad \bar{m}\bar{U} - \lambda\frac{\partial \bar{V}}{\partial Y} - \frac{1-\nu}{\nu}\alpha\frac{\partial \bar{W}}{\partial Z} = 0 \quad \text{at } Z = 0 \text{ and } 1. \tag{8}$$

3. SOLUTION METHOD

In order to formulate an eigenvalue problem from the governing differential equations and the relevant boundary conditions, eqns (4)–(8) are transformed into algebraic equations via the differential quadrature method. For this purpose, consider the quadrature grid shown in Fig. 2 having a set of $N_y \times N_z$ sampling points. In accordance with the DQM, the partial derivatives of a function $F(Y, Z)$ at a point Y_i, Z_j may be expressed by the following quadrature rules (Bert and Malik, 1996a)

$$\left.\frac{\partial^r F}{\partial Y^r}\right|_{Y_i,Z_j} = \sum_{k=1}^{N_y} A_{ik}^{(r)} F_{kj}, \quad \left.\frac{\partial^s F}{\partial Z^s}\right|_{Y_i,Z_j} = \sum_{l=1}^{N_z} B_{jl}^{(s)} F_{il}, \quad \left.\frac{\partial^{r+s} F}{\partial Y^r \partial Z^s}\right|_{Y_i,Z_j} = \sum_{k=1}^{N_y}\sum_{l=1}^{N_z} A_{ik}^{(r)} B_{jl}^{(s)} F_{kl} \tag{9}$$

where $F_{ij} = F(Y_i, Z_j)$, and $A_{ik}^{(r)}$ and $B_{jl}^{(s)}$ are the weighting coefficients of the partial derivatives with respect to the Y- and Z-coordinates, respectively. The weighting coefficients depend on the assumed form of the test functions and the distribution of the sampling points; the techniques of determining these coefficients may be found in the literature (Bert and Malik, 1996a).

Using the quadrature rules, eqn (9), in eqn (4), one obtains the quadrature analogs of the governing differential equations as

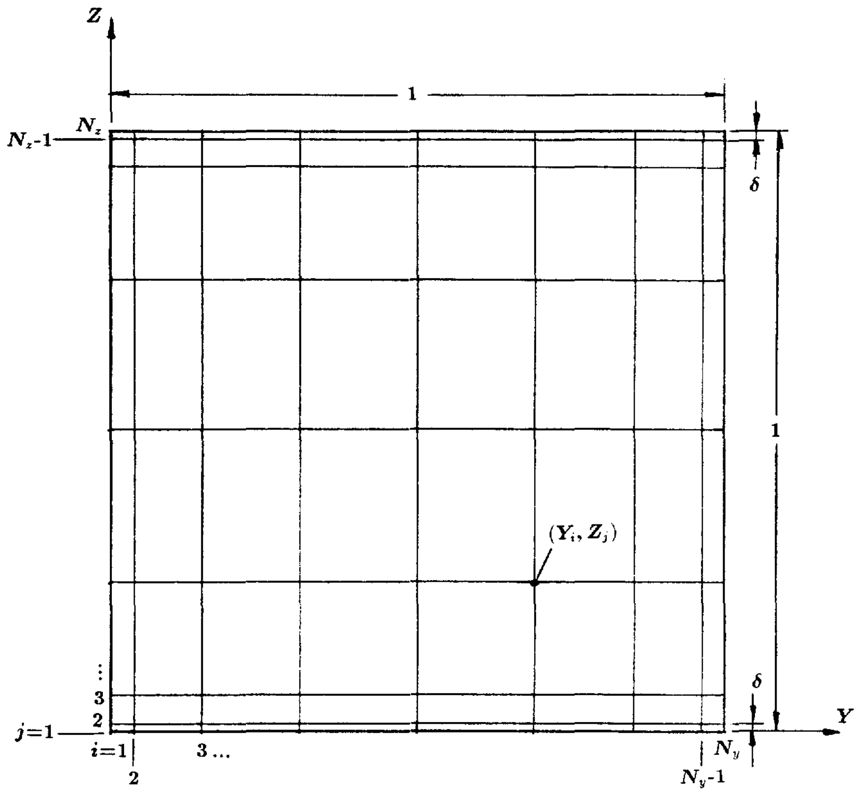

Fig. 2. A quadrature grid.

$$-\lambda^2 \sum_{k=1}^{N_y} A_{ik}^{(2)} \bar{U}_{kj} - \alpha^2 \sum_{l=1}^{N_z} B_{jl}^{(2)} \bar{U}_{il} + \frac{2(1-\nu)}{1-2\nu} \bar{m}^2 \bar{U}_{ij} - \frac{\bar{m}\lambda}{1-2\nu} \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{V}_{kj} - \frac{\bar{m}\alpha}{1-2\nu} \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{W}_{il} = \Omega^2 \bar{U}_{ij} \quad (10)$$

$$\frac{\bar{m}\lambda}{1-2\nu} \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{U}_{kj} - \frac{2(1-\nu)}{1-2\nu} \lambda^2 \sum_{k=1}^{N_y} A_{ik}^{(2)} \bar{V}_{kj} - \alpha^2 \sum_{l=1}^{N_z} B_{jl}^{(2)} \bar{V}_{il} + \bar{m}^2 \bar{V}_{ij} - \frac{\bar{m}\alpha\lambda}{1-2\nu} \sum_{k=1}^{N_y} \sum_{l=1}^{N_z} A_{ik}^{(1)} B_{jl}^{(1)} \bar{W}_{kl} = \Omega^2 \bar{V}_{ij} \quad (11)$$

$$\frac{\bar{m}\alpha}{1-2\nu} \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{U}_{il} - \frac{\bar{m}\alpha\lambda}{1-2\nu} \sum_{k=1}^{N_y} \sum_{l=1}^{N_z} A_{ik}^{(1)} B_{jl}^{(1)} \bar{V}_{kl} - \lambda^2 \sum_{k=1}^{N_y} A_{ik}^{(2)} \bar{W}_{kj} - \frac{2(1-\nu)}{1-2\nu} \alpha^2 \sum_{l=1}^{N_z} B_{jl}^{(2)} \bar{W}_{il} + \bar{m}^2 \bar{W}_{ij} = \Omega^2 \bar{W}_{ij} \quad (12)$$

where the ranges of indices i and j are:

$$i = 2, 3, \ldots, (N_y - 1); \quad j = 2, 3, \ldots, (N_z - 1)$$

i.e., the quadrature analogs of the governing differential equations are written at the interior points of the quadrature grid shown in Fig. 2 yielding $3(N_y - 2) \times 3(N_z - 2)$ linear equations. Additional equations to complete the set of $3N_y \times 3N_z$ linear equations are obtained from the quadrature analogs of the boundary conditions written at the boundary points of the quadrature grid.

The quadrature analogs of the boundary conditions on the lateral surfaces are written from eqn (8) as

$$\alpha \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{U}_{il} + \bar{m}\bar{W}_{ij} = 0, \quad \alpha \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{V}_{il} + \lambda \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{W}_{kj} = 0,$$

$$\bar{m}\bar{U}_{ij} - \lambda \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{V}_{kj} - \frac{1-\nu}{\nu} \alpha \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{W}_{il} = 0 \tag{13}$$

where $i = 1, 2, \ldots, N_y$ and, $j = 1$ and N_z for $Z = 0$ and 1, respectively. It is noted that these quadrature analog equations, eqn (13), are common to all types of plates being considered herein.

In writing the quadrature analogs of the boundary conditions of the y-sides, it is noted that at the corner points of the solution domain the conditions of both the y- and z-sides apply simultaneously. This situation may be taken care of by having one point close to each corner point at the ends of each of the two y-sides. As shown in Fig. 2, this is simply done be adding one grid line adjacent to each of the z-sides separated by a very small distance δ of the order 10^{-4} to 10^{-6} (on the normalized scale).

In order to illustrate the quadrature analog of the boundary conditions in the y-sides, consider a plate having the $Y = 0$ and 1 sides as simply supported and free, respectively. Then the quadrature analogs of the boundary conditions of the side $Y = 0$ may be written from eqn (5) as

$$\bar{U}_{ij} = 0, \quad \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{V}_{kj} = 0, \quad \bar{W}_{ij} = 0, \quad i = 1 \tag{14}$$

and similarly, from eqn (7), the quadrature analogs of the boundary conditions of the side $Y = 1$ may be written as

$$\lambda \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{U}_{kj} + \bar{m}\bar{V}_{ij} = 0, \quad \bar{m}\bar{U}_{ij} - \frac{1-\nu}{\nu} \lambda \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{V}_{kj} - \alpha \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{W}_{il} = 0,$$

$$\alpha \sum_{l=1}^{N_z} B_{jl}^{(1)} \bar{V}_{il} + \lambda \sum_{k=1}^{N_y} A_{ik}^{(1)} \bar{W}_{kj} = 0, \quad i = N_y \tag{15}$$

where $j = 2, 3, \ldots, N_z - 1$. It may be noted that due to the small value of δ, the lines $j = 2$ and $j = N_z - 1$ are very near to the boundary lines $j = 1$ and $j = N_z$, respectively. Consequently, in eqns (14) and (15), the boundary conditions of the Y-sides are invoked at points very close to the corner points.

For plate configurations with other types of boundary conditions on the y-sides, since eqn (13) remain common, only the set of quadrature analog equations for the boundary conditions of the y-sides will be different.

The eigenvalue equations of the problem may be constructed from the quadrature analog equations of the governing equations and the boundary conditions. The details of the same may be found in some other works of the present investigators (Bert and Malik; 1996a, b).

4. PURE SHEAR MODES OF VIBRATION

In case of the pure shear modes of vibration, the volume dilatation equals zero, i.e.,

$$\Theta = \frac{\partial U}{\partial X} + \lambda \frac{\partial V}{\partial Y} + \alpha \frac{\partial W}{\partial Z} = 0 \tag{16}$$

and consequently, eqn (1) is reduced to the following:

$$\nabla^2(U, V, W) = -\Omega^2(U, V, W). \tag{17}$$

It is noted that there is no coupling in these equations. However, the displacements in pure shear modes are indeed coupled; coupling comes through the boundary conditions.

Using eqn (3) in eqn (17), one obtains the governing equations of pure shear modes of vibration in terms of the reduced displacement variables as

$$\left\{\bar{m}^2 - \lambda^2 \frac{\partial^2}{\partial Y^2} - \alpha^2 \frac{\partial^2}{\partial Z^2}\right\}(\bar{U}, \bar{V}, \bar{W}) = \Omega^2(\bar{U}, \bar{V}, \bar{W}). \tag{18}$$

The boundary conditions on simply supported or clamped y-sides remain the same as given by eqns (5) and (6). However, on a free y-side, the condition of zero normal traction in eq (7) is simplified due to eqn (16). The boundary conditions on a free y-side may be written as

$$\lambda\frac{\partial \bar{U}}{\partial Y} + \bar{m}\bar{V} = 0, \quad \frac{\partial \bar{V}}{\partial Y} = 0, \quad \alpha\frac{\partial \bar{V}}{\partial Z} + \lambda\frac{\partial \bar{W}}{\partial Y} = 0 \quad \text{at } Y = 0 \text{ and/or } 1. \tag{19}$$

Similarly, the condition of zero normal traction in eqn (8) is simplified due to eqn (16). The boundary conditions on the lateral surfaces may be written as

$$\alpha\frac{\partial \bar{U}}{\partial Z} + \bar{m}\bar{W} = 0, \quad \alpha\frac{\partial \bar{V}}{\partial Z} + \lambda\frac{\partial \bar{W}}{\partial Y} = 0, \quad \frac{\partial \bar{W}}{\partial Z} = 0 \quad \text{at } Z = 0 \text{ and } 1. \tag{20}$$

The solutions of eqn (18) may be obtained consistent with the boundary conditions and the condition of zero dilatation, eqn (16), which, in terms of the reduced displacement variables, is

$$-\bar{m}\bar{U} + \lambda\frac{\partial \bar{V}}{\partial Y} + \alpha\frac{\partial \bar{W}}{\partial Z} = 0. \tag{21}$$

The possible forms of solutions for pure shear modes of vibration of six rectangular plate configurations having two opposite sides simply supported are given in Table 1. The

Table 1. Frequency equations for pure shear modes of free vibration of rectangular plates simply supported on two opposite sides and free on lateral surfaces

Plate	Modal displacement functions	Range of indices	Mode, frequency (f)
SSSS	$u = mA\cos m\pi X \sin n\pi Y \sin k\pi Z$ $v = n\lambda A \sin m\pi X \cos n\pi Y \sin k\pi Z$ $w = -k\alpha A \sin m\pi X \sin n\pi Y \cos k\pi Z$	$m, n = 1, 2, \ldots$ $k = h\sqrt{(m/a)^2 + (n/b)^2}$	Twist $\gamma_1\sqrt{m^2 + n^2\lambda^2}$
	$u = n\lambda A \cos m\pi X \sin n\pi Y \cos k\pi Z$ $v = -mA \sin m\pi X \sin n\pi Y \cos k\pi Z$ $w = 0$	$m, n, k = 0, 1, 2, \ldots$	Twist and torsion $\gamma_2\sqrt{m^2 + n^2\lambda^2 + k^2\alpha^2}$
SCSC, SSSC	$u = A \sin n\pi Y \cos k\pi Z$ $v = 0, w = 0$	$m = 0; 1, 2, \ldots$ $k = 0, 1, 2, \ldots$	Twist and torsion $\gamma_2\sqrt{n^2\lambda^2 + k^2\alpha^2}$
SSSF, SCSF	$u = A \sin(2n+1)(\pi/2) Y \cos k\pi Z$ $v = 0, w = 0$	$m = 0$ $n, k = 0, 1, 2, \ldots$	Twist and torsion $\gamma_2\sqrt{(2n+1)^2(\lambda/2)^2 + k^2\alpha^2}$
SSSF, SFSF	$u = \text{mA}\cos m\pi X \sin n\pi Y$ $v = -n\lambda A \sin m\pi X \cos n\pi Y, w = 0$	$m = n\lambda$ $m, n = 1, 2, \ldots$	Twist $\gamma_1 m$
SFSF	$u = mA\cos m\pi X \sin k\pi Z$ $v = 0, w = -k\alpha \sin m\pi X \cos k\pi Z$	$m = k\alpha$ $m, k = 1, 2, \ldots$	Torsion $\gamma_1 m$
	$u = A\cos n\pi Y \cos k\pi Z$ $v = 0, w = 0$	$m = 0$ $n, k = 0, 1, 2, \ldots$	Twist and torsion $\gamma_2\sqrt{n^2\lambda^2 + k^2\alpha^2}$

$\gamma_1 = \alpha\sqrt{3(1-\nu)}$, $\gamma_2 = \alpha\sqrt{3(1-\nu)/2}$.

pure shear modes of vibration may be of three types. With dilatation $\Theta = 0$ in each case, these modes are: a thickness-twist mode in which the transverse shear stresses $\tau_{yz} = \tau_{zx} = 0$, a torsional mode in which the in-plane shear stress $\tau_{xy} = 0$, and a coupled thickness-twist-torsional mode in which the in-plane shear and either or both of the transverse shear stresses are nonzero. In each of the solutions given in Table 1, such modes are identified.

It needs to be mentioned here that the solutions for simply supported (SSSS) plates given in Table 1 are actually included in the work of Srinivas *et al.* (1970a); these are included here for completeness.

5. RESULTS AND DISCUSSION

Based on the analysis of the foregoing sections, the natural frequencies of the six types of rectangular plates are obtained by the solution of three-dimensional elasticity equations. In all cases, the Poisson's ratio for the plate material is taken as $v = 0.3$. The results are obtained for combinations of three aspect ratios $a/b = 1/2$, 1, and 2, and two thickness ratios $h/b = 0.1$ and 0.2. The natural frequencies (f) being reported herein are in the following dimensionless forms which is consistent with commonly used form of dimensionless frequency in the plate literature.

$$f = \frac{\omega a^2}{2\pi}\sqrt{\frac{\rho h}{D}} = \frac{\alpha}{2\pi}\sqrt{6(1-v)}\Omega \tag{22}$$

where $D = Eh^3/[12(1-v^2)]$ is the flexural rigidity of the plate.

Two factors of importance in the convergence and accuracy of differential quadrature solutions are the weighting coefficients and the sampling points (Quan and Chang; 1989a, b; Bert and Malik, 1996a). The weighting coefficients are obtained most accurately by the explicit formulae of Quan and Chang (1989a) and the same are employed in the present work as well. Based on past experience of the present investigators with quadrature solutions of very many boundary-value and eigenvalue problems (Malik and Bert; 1996a, b), the sampling points used in the present work are given, with reference to Fig. 2, as

$$Y_i = \frac{1-\cos[(i-1)\pi/(N_y-1)]}{2}; \quad i = 1, 2, \ldots, N_y \tag{23}$$

and

$$Z_1 = 0, \quad Z_2 = \delta, \quad Z_{N_z-1} = 1-\delta, \quad Z_{N_z} = 1, \quad Z_{j+1} = \frac{1-\cos[(j-1)\pi/(N_z-3)]}{2};$$
$$j = 2, 3, \ldots, (N_z-3) \tag{24}$$

in the Y and Z directions, respectively.

The results of the present investigations are given in Tables 2–10 and Figs 3 and 4. Of these, the results of Tables 2–4 and Fig. 3 pertain to the convergence and accuracy of the quadrature solutions. First, Table 2 provides some sample results showing the convergence of the quadrature solution for a SSSS plate. The converged quadrature-solution values are compared with the exact values which are calculated using the characteristic frequency equation of Srinivas *et al.* (1970a). It is apparent that the frequencies obtained by the DQM solution actually match the exact values to at least as many as seven significant digits.

The SSSS plate is rather special since, in this case, the elasticity equations may be reduced to ordinary differential equations in the Z-coordinate; Srinivas *et al.* (1970a) derived the characteristic frequency equation from these equations only. These equations are given in Appendix 1 for completeness; the corresponding quadrature analogs of these

Table 2. Convergence of DQM solutions of reduced two-dimensional elasticity equations for natural frequencies (f) of SSSS plates

	Solution method				
	DQM solution: $N_z = 9$				Exact‡
	$a/b = 1/2, h/b = 0.2, m = 1, n = 3$				
N_y	12	13	14	15	
f	3.177907	3.177956	3.177944	3.177943†	3.177943
	$a/b = 1/2, h/b = 0.2, m = 2, n = 1$				
N_y	7	8	9	10	
f	3.834173	3.834170	3.834168	3.834169†	3.834169
	$a/b = 1, h/b = 0.1, m = 2, n = 2$				
N_y	11	12	13	14	
f	11.157342	11.157415	11.157411	11.157410†	11.157410
	$a/b = 1, h/b = 0.2, m = 1, n = 3$				
N_y	13	14	15	16	
f	10.503674	10.503597	10.503588	10.503590†	10.503590
	$a/b = 2, h/b = 0.1, m = 5, n = 1$				
N_y	8	9	10	11	
f	40.840046	40.839882	40.839885	40.839886†	40.839886
	$a/b = 2, h/b = 0.1, m = 3, n = 2$				
N_y	11	12	13	14	
f	35.678647	35.679032	35.679014	35.679007†	35.679009

† Converged value of DQM solution.

‡ Exact values calculated from the transcendental equation of Srinivas *et al.* (1970a); the same values are given by the DQM solution of reduced one-dimensional elasticity equations with $N_z = 9$.

Table 3. Effect of thickness ratio on the natural frequencies (f) of antisymmetric modes of rectangular plates having two opposite edges simply supported

Plate type	a/b	m, n	h/b ratio for elasticity solutions 0.2	0.1	0.02	0.01	0.005	CPT solution
SSSS	1/2	1, 1	1.531177	1.815133	1.956665	1.961786	1.962988	1.963495
SCSC	1	3, 2	12.99564	17.98121	22.06286	22.25712	22.30661	22.31424
SSSC	2	4, 1	25.03306	30.09690	32.77021	32.87303	32.89453	32.89686
SSSF	1/2	1, 4	3.481123	4.798997	5.833354	5.884730	5.898097	5.908996
SCSF	1	2, 1	5.270254	6.163496	6.598192	6.609886	6.610726	6.637068
SFSF	2	2, 3	18.62954	21.58862	23.19873	23.28252	23.29759	23.38027

equations are given in Appendix 2. Interestingly, the DQM solution of these one-dimensional elasticity equations gives results matching with the exact values to at least seven significant digits with as small as $N_z = 9$ sampling points.

Table 2 demonstrates fast convergence of the quadrature solutions for SSSS plates and that highly accurate results are obtainable with a moderate grid size with two-dimensional elasticity equations. However, for other types of boundary conditions, convergence is rather slow and fully converged solutions are obtained only with a larger number of grid points. As one particular example, the convergence of quadrature solutions for square SCSC plates is illustrated in Fig. 3, where each curve shows convergence of frequency values to at least five significant digits. As may be seen in this particular case, the quadrature solution seems to have reached its convergence for $N_y = 21$, $N_z = 17$. In fact this number of sampling points was found to be uniformly acceptable for all types of plates and all combinations of aspect and thickness ratios in that any increase in either N_y or N_z beyond this number of points added to the cost of computation for the frequency values changing only in the sixth or higher digits.

As mentioned earlier, the DQM solutions are highly accurate with respect to exact solutions of the SSSS plates. In fact, the DQM solutions have been found to exhibit the

Table 4. Some comparisons of elasticity solutions for natural frequencies (f) of square plates with different boundary conditions

Plate type	Solution method	Mode sequence number 1	4	7	1	4	7
		Thickness ratio, $h/b = 0.1$			Thickness ratio, $h/b = 0.2$		
SCSC	I	4.27479	—	16.3303	3.60265	7.22371	9.63249
	II	4.27068	10.2469	16.3143	3.60027	7.21765	9.63118
	III	4.26677	10.2470	16.2975	3.59831	7.21377	9.63060
SSSC	I	3.57635	—	12.1649	3.16086	6.25862	9.44315
	III	3.57358	10.2470	13.7437	3.15930	6.25826	9.44114
SSSF	I	1.81132	6.10610	8.95666	1.70613	4.47833	7.24737
	III	1.81362	6.10725	8.95479	1.70639	4.47695	7.24569
SCSF	I	1.95279	6.16395	10.9079	1.81447	5.26986	—
	III	1.95160	6.16350	10.9074	1.81408	5.27025	7.68521
SFSF	I	1.50312	5.80105	—	1.43264	4.65715	5.75635
	II	1.50341	5.79912	9.92131	1.43257	4.65741	5.75713
	III	1.50338	5.79912	9.92088	1.43258	4.65738	5.75717

I: Spline prism method (Mizusawa and Takagi, 1995); II: Ritz method (Liew *et al.*, 1993, 1995); III: DQM (present). Here, for consistent normalization, frequency values of Mizusawa and Takagi and Liew *et al.*, are multiplied by factors of $\alpha\sqrt{3(1-\nu^2)}/\pi$ and $\pi\lambda^2/2$, respectively.

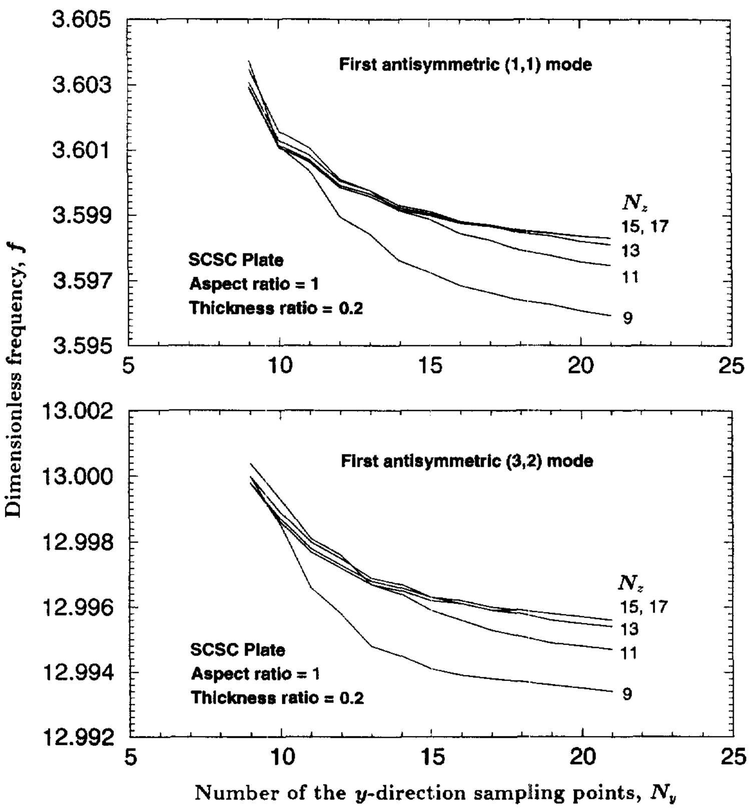

Fig. 3. Convergence of DQM solutions of reduced two-dimensional elasticity equations for natural frequencies (f) of SCSC plates.

same high level of accuracy with respect to the exact solutions of pure shear modes of all six types of plate configurations as given in Table 1. Such comparisons are *not* included here to avoid any repetitions and an overemphasis on the accuracy of the results. Nevertheless, some other assessment of accuracy of present results with respect to cases for which results are necessarily obtained by numerical means only, would be worthwhile. As an indirect evaluation of the accuracy of results of the present solution method, Table 3 shows some sample results of the frequencies of six types of plates (including the exact case of SSSS plate) for arbitrarily chosen modes and decreasing values of the thickness ratio (h/b);

Table 5. Natural frequencies (f) of the first nine modes of SSSS and SCSC plates from the three-dimensional elasticity theory solutions

	SSSS plates		SCSC plates	
	Thickness ratio (h/b)			
a/b	0.1	0.2	0.1	0.2
1/2	1.81513(1,1)a	1.28087(0,1)t	1.96322(1,1)a	1.28087(1,0)t
	2.56174(0,1)t	1.53118(1,1)a	2.56174(1,0)t	1.60289(1,1)a
	2.78935(1,2)a	2.21970(1,2)a	3.15971(1,2)a	2.36604(1,2)a
	4.27211(1,3)a	2.56174(1,0)t	4.77989(1,3)a	2.56174(2,0)t
	5.12348(1,0)t	2.86411(1,1)t	5.12348(2,0)t	3.04235(1,2)s
	5.35794(2,1)a	3.17794(1,3)a	5.39870(2,1)a	3.33291(1,3)a
	5.72822(1,1)t	3.62284(1,2)t	6.08250(1,2)s	3.84261(3,0)t
	6.12471(1,4)a	3.83417(2,1)a	6.25847(2,2)a	3.84607(2,1)a
	7.24569(1,2)t	3.84261(0,3)t	6.67886(1,4)a	3.87521(1,1)s
1	3.03828(1,1)a	2.78935(1,1)a	4.26677(1,1)a	3.59831(1,1)a
	7.26053(1,2)a	5.12348(1,0)t	7.85289(1,1)a	5.12348(1,0)t
	10.2470(1,0)t	6.12471(1,2)a	9.49717(1,2)a	6.41155(2,1)a
	11.1574(2,2)a	7.24569(1,1)t	10.2470(1,0)t	7.21377(1,2)a
	13.6058(1,3)a	8.87880(2,2)a	12.6388(2,2)a	9.05463(1,2)s
	14.4914(1,1)t	10.2470(2,0)t	13.9040(3,1)a	9.46414(2,2)a
	17.0884(2,3)a	10.5036(1,3)a	16.2975(1,3)a	9.63060(1,1)s
	20.4939(2,0)t	11.4564(1,2)t	17.9812(3,2)a	10.2470(2,0)t
	21.4318(1,4)a	12.1682(1,1)s	18.0874(1,2)a	10.6123(3,1)a
2	7.68882(1,1)a	7.26053(1,1)a	14.1590(1,1)a	12.0310(1,1)a
	12.1531(2,1)a	10.2470(1,0)t	17.0671(2,1)a	14.3932(2,1)a
	19.3681(3,1)a	11.1574(2,1)a	22.8011(3,1)a	19.0336(3,1)a
	20.4939(1,0)t	17.0884(3,1)a	31.4116(4,1)a	20.4939(1,0)t
	24.9535(1,2)a	20.4939(2,0)t	35.0682(1,2)a	25.6462(4,1)a
	29.0421(4,1)a	21.4318(1,2)s	37.9887(2,2)a	26.3189(1,1)s
	35.6790(3,2)a	22.9129(1,1)t	40.9878(1,0)t	26.7056(1,2)a
	40.8399(5,1)a	24.4988(4,1)a	42.5027(3,2)a	28.8551(2,2)a
	40.9878(2,0)t	28.9828(2,1)t	43.1119(3,2)a	32.5892(3,2)a

The letters a and s denote general modes in which in-plane displacement components are antisymmetric and symmetric, respectively about the mid-plane; the letter t denotes pure shear modes.

here the modes are chosen on the basis of classical theory solutions. As one would expect, it may be seen that with plates becoming thinner, the frequencies in all cases approach to the values of classical plate theory. Next, Table 4 shows a comparison of DQM solutions with the recently published results of Mizusawa and Takagi (1995) and Liew *et al.* (1993, 1995). The table includes frequencies of three modes of five types of square plates for two values of thickness ratios. It may be seen that there is a very close agreement between the three sets of results which are indeed by three different methods of solutions; of course, the DQM results are generally on a lower side than those of the other two solution methods.

Tables 5–7 provide natural frequencies of the first nine modes of the six types of plates each for six combinations of aspect and thickness ratios. Of these results, the frequency data of SSSS plates in Table 5 were obtained by quadrature solution of the reduced one-dimensional (ordinary) differential equations with $N_z = 9$. The frequency data of the remaining plates were obtained by the quadrature solution of the reduced two-dimensional differential equations with $N_y \times N_z = 21 \times 17$ grid points; on the basis of convergence studies, it is believed that the present calculations have produced results accurate to at least five significant digits, and for this reason, in all these tables, the frequencies are given to six significant digits. However, it has to be emphasized that in these tables, the frequency data of SSSS plates and of pure shear modes of all types of plates are exact.

It may be seen that the first nine mode frequencies of Tables 5–7 include general modes as well as the pure shear modes; in these tables, the two types of modes are appropriately labelled by two digits in parentheses followed by the letters a, s, or t. For the general modes, the two digits refer to the number of half-waves of the lateral displacement component w. The symbols a and s refer, respectively, to the asymmetry and symmetry of the in-plane displacement components (u and v) about the mid-plane. In fact, this identification has its

Table 6. Natural frequencies (f) of the first nine modes of SSSC and SSSF plates from the three-dimensional elasticity theory solutions

	SSSC plates		SSSF plates	
	Thickness ratio (h/b)			
a/b	0.1	0.2	0.1	0.2
1/2	1.88106(1,1)a	1.28087(1,0)t	1.28087(0,0)t	0.64063(0,0)t
	2.56174(1,0)t	1.56457(1,1)a	1.52681(1,1)a	1.31024(1,1)a
	2.96617(1,2)a	2.29210(1,2)a	2.12503(1,2)a	1.74787(1,2)a
	4.52279(1,3)a	2.56174(2,0)t	3.24030(1,3)a	1.92130(1,0)t
	5.12348(2,0)t	2.66417(1,1)s	3.84261(1,0)t	2.33568(1,1)s
	5.32814(1,1)s	3.25627(1,3)a	4.79900(1,4)a	2.50690(1,3)a
	5.37728(2,1)a	3.83997(2,1)a	5.10632(2,1)a	2.78214(1,1)s
	6.18931(2,1)a	3.84261(3,0)t	5.56477(1,1)s	3.20217(2,0)t
	6.40339(1,1)s	4.20374(1,1)s	5.60660(2,2)a	3.48112(1,4)a
1	3.57358(1,1)a	3.15930(1,1)a	1.81362(1,1)a	1.70639(1,1)a
	7.52422(2,1)a	5.12348(1,0)t	4.17139(1,2)a	2.56174(0,0)t
	8.34538(1,2)a	6.08246(1,1)s	5.12348(0,0)t	3.70138(1,2)a
	10.2470(1,0)t	6.25826(2,1)a	6.10725(2,1)a	4.47695(1,1)s
	11.8647(2,2)a	6.67851(1,2)a	8.50012(2,2)a	5.24095(2,1)a
	12.1622(1,1)a	9.16841(2,2)a	8.88040(1,3)a	6.99150(2,2)a
	13.7437(3,1)a	9.44114(1,1)s	8.95479(1,2)a	7.24569(1,1)t
	14.9516(1,3)a	10.2470(2,0)t	12.5282(3,1)a	7.27160(1,3)a
	17.5159(3,2)a	10.5557(3,1)a	12.9612(2,3)a	7.68521(1,0)t
2	10.5763(1,1)a	9.49506(1,1)a	2.52424(1,1)a	2.45092(1,1)a
	14.2943(2,1)a	12.6372(2,1)a	7.24504(2,1)a	6.82558(2,1)a
	20.8682(3,1)a	17.9800(3,1)a	11.5215(1,2)a	7.75970(1,1)s
	29.9190(1,2)a	18.0871(1,1)s	14.6102(3,1)a	10.2470(0,0)t
	30.0969(4,1)a	20.4939(1,0)t	15.5199(1,1)s	10.5692(1,2)a
	33.3815(2,2)a	24.1654(1,2)a	16.6745(2,2)a	13.1718(3,1)a
	36.1484(1,1)s	24.3298(2,1)s	20.4939(0,0)t	14.8055(2,2)a
	39.2461(3,2)a	25.0331(4,1)a	24.2647(3,2)a	17.9078(2,1)s
	40.9878(1,0)t	26.7140(2,2)a	24.4290(4,1)a	19.2867(1,1)s

The letters a and s denote general modes in which in-plane displacement components are antisymmetric and symmetric, respectively about the mid-plane; the letter t denotes pure shear modes.

basis in the classical and first-order shear deformable plate theories in which the in-plane displacements are taken to be first-order asymmetric while lateral displacement is taken to be uniform about the mid-plane. As it would be expected, in three-dimensional elasticity solutions, with respect to the mid-plane, the lateral displacement modes are found to be opposite to the modes of the in-plane displacements. In fact, for the first antisymmetric mode (1, 1)a, the lateral displacements happen to be close to uniform about the mid-plane. The symbol t denotes a pure shear mode. Referring to Table 1, the two digits associated with pure shear modes denote either the indices m and n for SSSS plates or the two indices in the expression of the u-displacement for the other types of plates.

The three-dimensional elasticity solutions can display an infinitely unlimited number of frequencies for any vibration mode. However, the classical and first- and higher-order shear deformable plate theories can display only a limited number of frequencies for each mode of vibration. Of course, this limitation comes due to the degree of approximation in the assumed form of displacement functions. Thus, the CPT gives only one frequency of the first antisymmetric mode of vibration. The FSDPT defines the flexural mode by three variables (two cross-sectional rotations in addition to w) and assumes the same linear form of in-plane displacements (u and v) with respect to the through-thickness z-coordinate. Consequently, the FSDPT yields a set of three frequencies for an antisymmetric mode of vibration. However, it should be noted that neither the CPT or FSDPT can display symmetric modes of vibration. The HSDPT can display a larger number of frequencies depending on the number of variables assumed to describe the flexural mode of vibration. Also, a higher-order theory can display both antisymmetric and symmetric modes depending on the assumed variation of the displacements with respect to the z-coordinate. As some specific examples, the higher-order theory of Reddy (Reddy, 1984; Reddy and Phan,

Table 7. Natural frequencies (f) of the first nine modes of SCSF and SFSF plates from the three-dimensional elasticity theory solutions

	SCSF plates		SFSF plates	
	Thickness ratio (h/b)			
a/b	0.1	0.2	0.1	0.2
1/2	1.28087(0,0)t	0.64043(0,0)t	1.44978(1,1)a	1.25225(1,1)a
	1.54087(1,1)a	1.31756(1,1)a	1.70641(1,2)a	1.28087(1,0)t
	2.22912(1,2)a	1.79631(1,2)a	2.48022(1,3)a	1.43929(1,2)a
	3.43826(1,3)a	1.92130(1,0)t	2.56174(1,0)t	1.98683(1,3)a
	3.84261(1,0)t	2.36888(1,1)s	3.70144(1,4)a	2.23764(1,2)s
	4.73966(1,1)s	2.58114(1,3)a	4.47695(1,2)s	2.56174(2,0)t
	5.05581(1,4)a	3.20217(2,0)t	5.04582(2,1)a	2.57049(1,1)s
	5.10952(2,1)a	3.31632(1,2)s	5.12348(2,0)t	2.79030(1,4)a
	5.64240(2,2)a	3.55423(1,4)a	5.14376(1,1)s	3.62284(1,2)t
1	1.95160(1,1)a	1.81408(1,1)a	1.50338(1,1)a	1.43258(1,1)a
	4.85573(1,2)a	2.56174(0,0)t	2.45096(1,2)a	2.24773(1,2)a
	5.12348(0,0)t	4.13040(1,2)a	5.39715(1,3)a	3.87942(1,2)s
	6.16350(2,1)a	5.27025(2,1)a	5.79912(2,1)a	4.65738(1,3)a
	8.91649(2,2)a	5.45469(1,1)s	6.82565(2,2)a	5.00899(2,1)a
	10.0383(1,3)a	7.18526(2,2)a	7.75970(1,2)s	5.12348(1,0)t
	10.9074(1,1)s	7.68521(1,0)t	9.92088(2,3)a	5.75717(2,2)a
	12.5542(3,1)a	7.81288(1,3)a	10.2470(1,0)t	7.24569(1,1)t
	13.7530(2,3)a	8.43511(1,1)s	10.5695(1,4)a	7.94732(2,3)a
2	3.54166(1,1)a	3.37696(1,1)a	1.50530(1,1)a	1.48500(1,1)a
	7.80640(2,1)a	7.25634(2,1)a	4.21244(1,2)a	3.97004(1,2)a
	14.9092(1,2)a	10.2470(0,0)t	6.01351(2,1)a	5.66776(1,2)s
	14.9555(3,1)a	12.9325(1,2)a	9.80383(2,2)a	5.73034(2,1)a
	19.4229(2,2)a	13.3900(3,1)a	11.3358(2,2)s	8.99092(2,2)s
	20.4939(0,0)t	16.5216(2,2)a	13.3483(3,1)a	12.1376(3,1)a
	24.6540(4,1)a	17.6727(1,1)s	15.9201(1,3)a	14.2662(1,3)a
	26.4229(3,2)a	19.4008(1,1)s	17.4670(3,2)a	15.4041(3,2)a
	35.3058(1,1)s	21.0810(4,1)a	21.5886(2,3)a	15.5177(2,2)a

The letters a and s denote general modes in which in-plane displacement components are antisymmetric and symmetric, repsectively about the mid-plane; the letter t denotes pure shear modes.

1985) has z and z^3 terms in the assumed form of u- and v-functions keeping the lateral displacement w independent of the z-coordinate. Thus, this theory can display only the antisymmetric modes of vibration. Another higher-order theory due to Cho *et al.* (1991) has z, z^2, and z^3 terms in u- and v-functions and z and z^2 terms in w-function. Consequently, Cho *et al.* (1991) could obtain some of the symmetric mode frequencies along with some antisymmetric modes.

As mentioned earlier, an exact three-dimensional elasticity solution for free vibration of SSSS plates was reported by Srinivas *et al.* (1970a). This work contained some extensive numerical results wherein the first nine mode frequencies were given for a set of values of the parameter $\sqrt{(mh/a)^2+(nh/b^2)}$. As may be noted, in the present work, the frequencies are given in a conventional format with respect to the geometry of the plates. From this point of view, the results of the SSSS plates given in the present paper should also be of interest to the readers.

In Tables 5–7, the frequencies are sorted as of the first nine modes. Of course, these frequencies can not be compared directly with the first nine mode frequencies based upon the approximate theories. However, a meaningful quantitative comparison can be made on the basis of similar modes of vibration. Such comparisons are given in Tables 8 and 9 on the basis of the first, fifth, and ninth mode shapes of the CPT for all six types of plates with three aspect ratios $a/b = 1/2$, 1, and 2, and two thickness ratios $h/b = 0.1$ and 0.2. It should be noted that the first, fifth, and ninth mode frequencies of the CPT are not necessarily the same order frequencies from either FSDPT or elasticity solutions. Needless to say, the frequencies from the CPT and FSDPT solutions are also obtained by the DQM and that the accuracies of these solutions have been verified by comparisons with the available solutions of the other investigators, for example with those of Leissa (1973) and Mizusawa

Table 8. Comparison of natural frequencies (f) from classical (CPT), first-order shear deformable (FSDPT), and three-dimensional elasticity (ELAST) solutions for SSSS, SCSC, and SSSC plates

Plate	a/b	CPT mode sequence #	m, n	CPT	FSDPT with Shear Correction Factors $\pi^2/12$	5.6	$5/(6-\nu)$	ELAST
					Thickness ratio, $h.b$ = 0.1			
SSSS	1/2	1	1, 1	1.96350	1.80831	1.80970	1.81497	1.81513
		5	1, 4	7.85398	6.05953	6.07211	6.12034	6.12471
	1	5	2, 3	20.4204	16.9525	16.9792	17.0816	17.0884
		9	3, 4	39.2699	28.9495	29.0177	29.2800	29.3073
	2	1	1, 1	7.85398	7.68065	7.68234	7.68877	7.68882
		9	1, 3	58.1195	50.4873	50.5500	50.7893	50.8016
SCSC	1/2	1	1, 1	2.17816	1.95214	1.95414	1.96179	1.96322
		5	2, 2	8.22422	6.18479	6.19861	6.25176	6.25847
	1	5	3, 1	16.2682	13.8020	13.8216	13.8969	13.9040
		9	4, 1	27.1115	21.3802	21.4215	21.5798	21.5948
	2	1	1, 1	15.1615	14.0649	14.0767	14.1216	14.1590
		9	1, 2	40.4473	34.7387	34.7917	34.9947	35.0682
SSSC	1/2	1	1, 1	2.05605	1.87257	1.87421	1.88045	1.88106
		5	2, 2	8.02637	6.12010	6.13327	6.18381	6.18931
	1	5	3, 1	15.9584	13.6490	13.6676	13.7387	13.7437
		9	4, 1	26.8906	21.2997	21.3402	21.4956	21.5091
	2	1	1, 1	11.0337	10.5394	10.5447	10.5647	10.5763
		9	4, 2	54.8349	47.1125	47.1768	47.4228	47.4588
					Thickness ratio, h/b = 0.2			
SSSS	1/2	1	1, 1	1.96350	1.51488	1.51803	1.53009	1.53118
		5	1, 4	7.85398	4.18860	4.20512	4.26920	4.28098
	1	5	2, 3	20.4204	12.4832	12.5250	12.6868	12.7118
		9	3, 4	39.2699	19.3763	19.4584	19.7776	19.8407
	2	1	1, 1	7.85398	7.23325	7.23879	7.25989	7.26053
		9	1, 3	58.1195	39.0397	39.1522	39.5859	39.6426
SCSC	1/2	1	1, 1	2.17816	1.58183	1.58568	1.60052	1.60289
		5	2, 2	8.22422	4.21853	4.23558	4.30182	4.31656
	1	5	3, 1	16.2682	10.4354	10.4677	10.5923	10.6123
		9	4, 1	27.1115	15.0676	15.1242	15.3437	15.3843
	2	1	1, 1	15.1615	11.8579	11.8857	11.9928	12.0310
		9	1, 2	40.4473	26.1888	26.2782	26.6252	26.7056
SSSC	1/2	1	1, 1	2.05605	1.54620	1.54965	1.56294	1.56457
		5	2, 2	8.02637	4.20341	4.22018	4.28532	4.29854
	1	5	3, 1	15.9584	10.3845	10.4160	10.5378	10.5557
		9	4, 1	26.8906	15.0466	15.1029	15.3209	15.3599
	2	1	1, 1	11.0337	9.40897	9.42359	9.47956	9.49506
		9	4, 2	54.8349	36.0751	36.1830	36.5999	36.6737

(1993); these comparisons were reported in other works of the present investigators (Malik and Bert, 1995; Bert and Malik, 1996c). For the FSDPT solutions, one needs a value of the shear correction factor (SCF). Two commonly used values of the factor are 5/6 from Reissner's work (1945) and $\pi^2/12$ from Mindlin's work (1951). The FSDPT solutions in Tables 8 and 9 are based on these two SCF values and a third value of $5/(6-\nu)$ which was *derived* by Wittrick (1987) through some very detailed analytical investigations on the exact solutions of elasticity equations and Mindlin theory for simply supported plates.

Two other comparisons of the three types of solutions for the frequency of the first antisymmetric (1, 1) mode of the six types of plates are given in Fig. 4 and Table 10. In Fig. 4, the frequencies of square plates are plotted against thickness ratio h/b ranging from 0.01 to 1; here the FSDPT frequencies are based on SCF value of $\pi^2/12$. In Table 10, the frequencies are for cubic parallelepipeds; here the FSDPT frequencies are based on all three SCF values and ratios of the CPT and FSDPT frequencies with respect to the corresponding elasticity values are included.

Table 9. Comparison of natural frequencies (f) from classical (CPT), first-order shear deformable (FSDPT), and three-dimensional elasticity (ELAST) solutions for SSSF, SCSF, and SFSF plates

Plate	a/b	CPT Mode sequence #	m, n	CPT	FSDPT with Shear Correction Factors $\pi^2/12$	$5/6$	$5/(6-\nu)$	ELAST
				Thickness ratio, $h/b = 0.1$				
SSSF	1/2	1	1, 1	1.63916	1.52234	1.52335	1.52723	1.52681
		5	2, 1	6.32960	5.06238	5.07153	5.10658	5.10632
	1	5	1, 3	9.84542	8.84520	8.85274	8.88146	8.88040
		9	1, 4	18.4120	15.4736	15.4945	15.5745	15.5775
	2	1	1, 1	2.56793	2.52340	2.52373	2.52497	2.52424
		9	4, 2	37.6023	33.8674	33.8976	34.0127	34.0005
SCSF	1/2	1	1, 1	1.65927	1.53614	1.53721	1.54128	1.54087
		5	1, 4	6.46055	5.00471	5.01456	5.05236	5.05581
	1	5	1, 3	11.5224	9.97085	9.88505	10.0301	10.0383
		9	3, 1	14.4212	12.4824	12.4979	12.5570	12.5542
	2	1	1, 1	3.63119	3.53759	3.53838	3.54140	3.54166
		9	5, 1	40.5399	36.3706	36.4055	36.5384	36.5289
SFSF	1/2	1	1, 1	1.54957	1.44568	1.44660	1.45010	1.44978
		5	2, 1	6.23698	5.00254	5.01151	5.04584	5.04582
	1	5	2, 2	7.43861	6.80554	6.81066	6.83021	6.82565
		9	3, 2	15.2853	13.0989	13.1158	13.1800	13.1719
	2	1	1, 1	1.51395	1.50502	1.50510	1.50540	1.50530
		9	4, 1	24.7931	23.1309	23.1456	23.2016	23.1965
				Thickness ratio, $h/b = 0.2$				
SSSF	1/2	1	1, 1	1.63916	1.29933	1.30172	1.31087	1.31024
		5	2, 1	6.32960	3.61116	3.62409	3.67416	3.67655
	1	5	1, 3	9.84542	7.19540	7.21054	7.26863	7.27160
		9	1, 4	18.4120	11.5511	11.5848	11.7145	11.7307
	2	1	1, 1	2.56793	2.44840	2.44928	2.45262	2.45092
		9	4, 2	37.6023	27.6788	27.7401	27.9756	27.9660
SCSF	1/2	1	1, 1	1.65927	1.30635	1.30880	1.31818	1.31756
		5	1, 4	6.46055	3.48469	3.49709	3.54520	3.55423
	1	5	1, 3	11.5224	7.69856	7.71944	7.80011	7.81288
		9	3, 1	14.4212	9.64854	9.67582	9.78095	9.78137
	2	1	1, 1	3.63119	3.36336	3.36556	3.37396	3.37696
		9	5, 1	40.5399	29.4528	29.5226	29.7912	29.7839
SFSF	1/2	1	1, 1	1.54957	1.24200	1.24421	1.25268	1.25225
		5	2, 1	6.23698	3.57633	3.58910	3.63853	3.64142
	1	5	2, 2	7.43861	5.71215	5.72321	5.76565	5.75717
		9	3, 2	15.2853	10.0405	10.0691	10.1791	10.1691
	2	1	1, 1	1.51395	1.48396	1.48423	1.48526	1.48500
		9	4, 1	24.7931	19.8720	19.9074	20.0428	20.0360

The results in Fig. 4. and Tables 8–10 are in conformity with the well known fact that the CPT overestimates the vibration frequencies quite in excess. In fact, rather commonly accepted norms of using CPT for plate having the thickness less than one-tenth or one-twentieth of the smaller side ($h/b < 1/10$ or $1/20$) does not seem to be adequate. The FSDPT frequencies with SCF values of $\pi^2/12$ and $5/6$ are close underestimates of the three-dimensional elasticity values for all six types of plate configurations; of the two factors, Reissner's value seems to be a better choice. Most interestingly, it is noted from Tables 8–10 that in all the cases, the FSDPT frequencies with Wittrick's shear correction factor match most closely with the elasticity values. It is also noted that with Wittrick's SCF value, the FSDPT frequencies remain lower than the elasticity values for plates without any free (vertical) sides, i.e. SSSS, SCSC, and SSSC plates. On the other hand, in some cases of plates with free (vertical) sides, i.e., SSSF, SCSF, and SFSF plates, particularly for $h/b = 0.1$, the FSDPT frequencies with Wittrick's SCF value are slightly higher than the elasticity values.

Table 10. Comparison of the first antisymmetric mode frequencies from classical (CPT), first-order shear deformable (FSDPT), and three-dimensional elasticity (ELAST) solutions for six types of cubic parallelepipeds

Cube	CPT	FSDPT with Shear Correction Factors $\pi^2/12$	$5/6$	$5/(6-\nu)$	ELAST
SSSS	3.14159 (2.574)	1.18569 (0.9715)	1.19169 (0.9764)	1.21514 (0.9956)	1.22050
SCSC	4.60767 (3.660)	1.21012 (0.9613)	1.21659 (0.9664)	1.24200 (0.9866)	1.25886
SSSC	3.76343 (3.091)	1.19406 (0.9807)	1.20021 (0.9856)	1.22433 (1.006)	1.21755
SSSF	1.85965 (2.159)	0.84860 (0.9850)	0.85195 (0.9889)	0.86493 (1.004)	0.86150
SCSF	2.01926 (2.294)	0.86492 (0.9828)	0.86846 (0.9868)	0.88219 (1.002)	0.88010
SFSF	1.53288 (1.976)	0.76151 (0.9816)	0.76458 (0.9856)	0.77650 (1.001)	0.77575

A value in parentheses is the ratio of the frequency value above it with the corresponding frequency from elasticity solution.

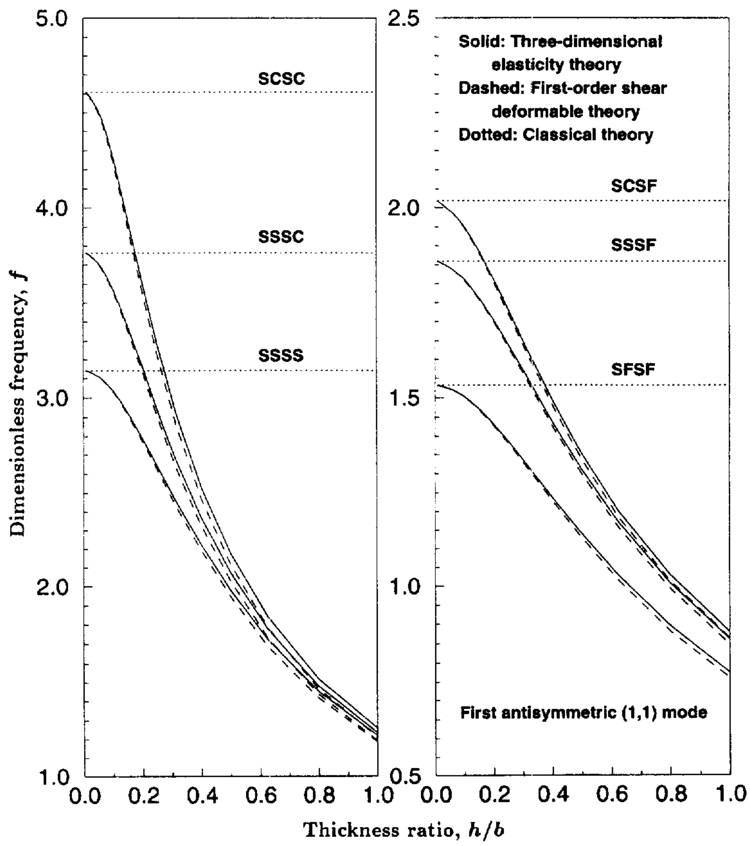

Fig. 4. A comparison of first antisymmetric mode frequencies from classical and first-order shear deformable theories and three-dimensional elasticity solutions for six types of plates (SCF = $\pi^2/12$ for FSDPT solutions).

In their work on simply supported plates, Srinivas *et al.* (1970a) compared the frequencies from the CPT and FSDPT solutions with the exact values of the elasticity solution. Srinivas *et al.* (1970a) pointed that the FSDPT frequencies, which were based on SCF value of $\pi^2/12$, were slightly lower than the exact values and that with a SCF value of 0.88, the FSDPT values can be made 'equal' to the elasticity values. Here, the FSDPT frequencies with SCF = 0.88 are not included for any further comparisons. However, it needs to be mentioned here that SCF = 0.88 does not really make the FSDPT frequencies equal to the exact values even for SSSS plates; it gives frequencies slightly higher than those obtained with Wittrick's SCF value. Thus, compared to the Wittrick's factor, SCF = 0.88 yields FSDPT frequencies closer to the exact values for plates without free sides and farther away from the exact values for plates with free sides. Wittrick's formula, which was based on an analysis of simply supported plates, shows dependence of the shear correction factor on Poisson ratio. It seems that the shear correction factor should actually depend on the boundary conditions as well as the geometric proportions of aspect ratio a/b and thickness ratio h/b.

Before concluding this work, two last points are in order. First, it should be remarked that the fundamental frequency of rectangular plates from the classical and first-order shear deformable theories correspond to those of the first antisymmetric mode (1, 1)a. However, as may be seen from the $a/b = 1/2$ frequencies of the SSSS, SCSC, and SCSF plates in Tables 5–7, the fundamental frequency may actually be from the pure shear modes. Next, it has to be realized that frequencies are *scanned* by trials. It is believed that the frequencies in Tables 5–7 are the first nine mode frequencies; the responsibility of any missing link is entirely of the present investigators. It should be pointed out that some omissions do exist in previous works, and these have been taken care of in the present work. As a specific example, besides some other missing values, Mizusawa and Takagi (1995) have not included pure shear mode frequencies in their results (this is the reason for some omissions in Table 4). Similarly some omissions have been found in frequency data of SSSS plates in the work of Liew *et al.* (1993, 1995) who have presented frequencies of twelve modes of various types of plates.

6. CONCLUDING REMARKS

The work for this paper was carried out with the objective of providing three-dimensional elasticity solutions for the free vibrations of rectangular plates. The plates considered were of the type having free lateral surfaces and two opposite sides having combinations of simply supported, clamped, and free boundary conditions. However, the boundary conditions at the other two opposite sides were taken to be simply supported; this restriction facilitated dimensional reduction in the governing equations from three to two and thereby, the computational effort was considerably eased. With these limited boundary conditions, the total number of plate configurations analyzed was six.

The numerical solution technique employed in the present work was the differential quadrature method. The frequency data contained in this paper were prepared after extensive convergence studies and validations for the accuracy of the results. The investigators believe that the results produced in this paper are of very high accuracy and that there are no missing links in mode sequences included in the results.

Acknowledgements—Some of the recent works on three-dimensional elasticity solutions were brought to notice by an anonymous reviewer; these helped greatly in giving up-to-date information in the paper.

REFERENCES

Bellman, R. and Casti, J. (1971) Differential quadrature and long-term integration. *Journal of Mathematical Analysis and Applications* **34,** 235–238.

Bellman, R., Kashef, B. G. and Casti, J. (1972) Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. *Journal of Computational Physics* **10,** 40–52.

Bert, C. W. and Malik, M. (1996a) Differential quadrature method in computational mechanics: A review. *Applied Mechanics Reviews* **49,** 1–28.

Bert, C. W. and Malik, M. (1996b) Free vibration analysis of thin cylindrical shells by the differential quadrature method. *Journal of Pressure Vessel Technology* **118,** 1–12.

Bert, C. W. and Malik, M. (1996c) Free vibration analysis of tapered rectangular plates by the differential quadrature method: A semi-analytical approach. *Journal of Sound and Vibration* **190,** 41–63.

Cheung, Y. K. and Chakrabarti, S. (1972) Free vibration of thick, layered rectangular plates by a finite layer method. *Journal of Sound and Vibration* **21,** 277–284.

Cho, K. N., Bert, C. W. and Striz, A. G. (1991) Free vibrations of laminated rectangular plates by higher order individual-layer theory. *Journal of Sound and Vibration* **145,** 429–442.

Fromme, J. A. and Leissa, A. W. (1970) Free vibration of the rectangular parallelepiped. *Journal of the Acoustical Society of America* **48,** 290–298.

Hutchinson, J. R. and Zillmer, S. D. (1983) Vibration of a free rectangular parallelepiped. *Journal of Applied Mechanics* **50,** 123–130.

Iyengar, K. T. S. R. and Raman, P. V. (1980) Free vibration of rectangular plates of arbitrary thickness with one or more edges clamped. *Journal of Sound and Vibration* **71,** 463–472.

Leissa, A. W. (1973) The free vibration of rectangular plates. *Journal of Sound and Vibration* **31,** 257–293.

Liew, K. M., Hung, K. C. and Lim, M. K. (1993) A continuum three-dimensional vibration analysis of thick rectangular plates. *International Journal of Solids and Structures* **30,** 3357–3379.

Liew, K. M., Hung, K. C. and Lim, M. K. (1994) Three-dimensional vibration of rectangular plates: Variance of simple support conditions and influence of in-plane inertia. *International Journal of Solids and Structures* **31,** 3233–3247.

Liew, K. M., Hung, K. C., and Lim, M. K. (1995) Three-dimensional vibration of rectangular plates: Effect of thickness and edge constraints. *Journal of Sound and Vibration* **182,** 709–727.

Malik, M. and Bert, C. W. (1995) Differential quadrature analysis of free vibration of symmetric cross-ply laminates with shear deformation and rotatory inertia. *Shock and Vibration* **2,** 321–338.

Mindlin, R. D. (1951) Influence of rotary inertia and shear in flexural motion of isotropic, elastic plates. *Journal of Applied Mechanics* **18,** 31–38.

Mizusawa, T. (1993) Vibration of rectangular Mindlin plates with tapered thickness by the spline strip method. *Computers and Structures* **46,** 451–463.

Mizusawa, T. and Takagi, S. (1995) Application of the spline prism method to analyze vibration of thick rectangular plates with two opposite edges simply supported. *Communications on Numerical Methods in Engineering* **11,** 1–11.

Noor, A. K. (1973a) Mixed finite-difference scheme for analysis of simply supported thick plates. *Computers and Structures* **3,** 967–982.

Noor, A. K. (1973b) Free vibrations of multilayered composite plates. *AIAA Journal* **11,** 1038–1039.

Noor, A. K. (1975) Stability of multilayered composite plates. *Fibre Science and Technology* **8,** 81–89.

Pagano, N. J. (1969) Exact solutions for composite laminates in cylindrical bending. *Journal of Composite Materials* **3,** 398–411.

Pagano, N. J. (1970) Exact solutions for rectangular bidirectional composites and sandwich plates. *Journal of Composite Materials* **4,** 20–34.

Quan, J. R. and Chang, C. T. (1989a) New insights in solving distributed system equations by the quadrature method—I. Analysis. *Computers in Chemical Engineering* **13,** 779–788.

Quan, J. R. and Chang, C. T. (1989b) New insights in solving distributed system equations by the quadrature method—II. Numerical experiments. *Computers in Chemical Engineering* **13,** 1017–1024.

Reddy, J. N. (1984) A simply higher-order theory for laminated composite plates. *Journal of Applied Mechanics* **51,** 745–752.

Reddy, J. N. and Phan, N. D. (1985) Stability and vibration of isotropic, orthotropic, and laminated plates according to a higher-order shear deformation theory. *Journal of Sound and Vibration* **98,** 157–170.

Reissner, E. (1945) The effect of transverse shear deformation on the bending of elastic plates. *ASME Journal of Applied Mechanics* **12,** A69–A77.

Srinivas, S., Rao, C. V. J. and Rao, A. K. (1970a) An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates. *Journal of Sound and Vibration* **12,** 187–199.

Srinivas, S., Rao, C. V. J. and Rao, A. K. (1970b) Some results from an exact analysis of thick laminates in vibration and buckling. *Journal of Applied Mechanics* **37,** 868–870.

Srinivas, S. and Rao, A. K. (1970c) Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. *International Journal of Solids and Structures* **6,** 1463–1481.

Wittrick, W. H. (1987) Analytical, three-dimensional elasticity solutions to some plate problems, and some observations on Mindlin's plate theory. *International Journal of Solids and Structures* **23,** 441–464.

Young, P. G., Yuan, J. and Dickinson, S. M. (1996) Three-dimensional analysis of free vibration of thick rectangular plates with depressions, grooves or cut-outs. *Journal of Vibration and Acoustics* **118,** 184–189.

APPENDIX 1

Here, the case of a SSSS plate is considered. The displacement components expressed in the following form satisfy the boundary conditions of both x- and y-sides.

$$U = \hat{U}(Z)\cos\bar{m}X\sin\bar{n}Y, \quad V = \hat{V}(Z)\sin\bar{m}X\cos\bar{n}Y, \quad W = \hat{W}(Z)\sin\bar{m}X\sin\bar{n}Y \tag{25}$$

where $\bar{n} = n\pi$ and n is an integer and $\hat{U}(Z)$, $\hat{V}(Z)$, and $\hat{W}(Z)$ are reduced displacement variables.

Using eqn (25) in eqn (1), one obtains

$$-\left\{\alpha^2\frac{d^2}{dZ^2} - \frac{2(1-\nu)}{1-2\nu}\bar{m}^2 - \bar{n}^2\lambda^2\right\}\hat{U} + \frac{\bar{m}\bar{n}\lambda}{1-2\nu}\hat{V} - \frac{\bar{m}\alpha}{1-2\nu}\frac{d\hat{W}}{dZ} = \Omega^2\hat{U}$$

$$\frac{\bar{m}\bar{n}\lambda}{1-2\nu}\hat{U} - \left\{\alpha^2\frac{d^2}{dZ^2} - \bar{m}^2 - \frac{2(1-\nu)}{(1-2\nu)}\bar{n}^2\lambda^2\right\}\hat{V} - \frac{\bar{n}\alpha\lambda}{1-2\nu}\frac{d\hat{W}}{dZ} = \Omega^2\hat{V}$$

$$\frac{\bar{m}\alpha}{1-2\nu}\frac{d\hat{U}}{dZ} + \frac{\bar{n}\alpha\lambda}{1-2\nu}\frac{d\hat{V}}{dZ} - \left\{\frac{2(1-\nu)}{1-2\nu}\alpha^2\frac{d^2}{dZ^2} - \bar{m}^2 - \bar{n}^2\lambda^2\right\}\hat{W} = \Omega^2\hat{W}. \tag{26}$$

The boundary conditions of the above equations are obtained by using eqn (25) in eqn (8) as

$$\alpha\frac{d\hat{U}}{dZ}+\bar{m}\hat{W}=0,\quad \alpha\frac{d\hat{V}}{dZ}+\bar{n}\lambda\hat{W}=0,\quad \bar{m}\hat{U}+\bar{n}\lambda\hat{V}-\frac{1-v}{v}\alpha\frac{d\hat{W}}{dZ}=0\quad \text{at } Z=0 \text{ and } 1. \tag{27}$$

It is noted that the three-dimensional elasticity equations are reduced to one-dimensional equations described in a line domain $0 \leqslant Z \leqslant 1$.

APPENDIX 2

Using the quadrature rules, one obtains the quadrature analogs of eqn (26) as

$$-\alpha^2\sum_{k=1}^{N_z}B_{ik}^{(2)}\hat{U}_k+\left[\frac{2(1-v)}{1-2v}\bar{m}^2+\bar{n}^2\lambda^2\right]\hat{U}_i+\frac{\bar{m}\bar{n}\lambda}{1-2v}\hat{V}_i-\frac{\bar{m}\alpha}{1-2v}\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{W}_k=\Omega^2\hat{U}_i$$

$$\frac{\bar{m}\bar{n}\lambda}{1-2v}\hat{U}_i-\alpha^2\sum_{k=1}^{N_z}B_{ik}^{(2)}\hat{V}_k+\left[\bar{m}^2+\frac{2(1-v)}{(1-2v)}\bar{n}^2\lambda^2\right]\hat{V}_i-\frac{\bar{n}\alpha\lambda}{1-2v}\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{W}_k=\Omega^2\hat{V}_i$$

$$\frac{\bar{m}\alpha}{1-2v}\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{U}_k+\frac{\bar{n}\alpha\lambda}{1-2v}\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{V}_k-\frac{2(1-v)}{1-2v)}\alpha^2\sum_{k=1}^{N_z}B_{ik}^{(2)}\hat{W}_k+[\bar{m}^2+\bar{n}^2\lambda^2]\hat{W}_i=\Omega^2\hat{W}_i \tag{28}$$

where $i = 2, 3, \ldots, N_z-1$. The boundary conditions are implemented by writing the quadrature analogs of eqn (27) at the boundary points $i = 1$ and N_z. These are

$$\alpha\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{U}_k+\bar{m}\hat{W}_i=0,\quad \alpha\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{V}_k+\bar{n}\lambda\hat{W}_i=0,\quad \bar{m}\hat{U}_i+\bar{n}\lambda\hat{V}_i-\frac{1-v}{v}\alpha\sum_{k=1}^{N_z}B_{ik}^{(1)}\hat{W}_k=0 \tag{29}$$

where $i = 1$ and N_z.